Final Exam, Group Theory

Block 1A, 2026–2027
October 27, 2026, 11:45-13:45

university of groningen

INSTRUCTIONS TO CANDIDATES

1. **Attempt all the questions in this test. You will get 10 points for free. The total number of points available is 100.**
2. **The number of points you can get for each question is shown next to it.**
3. **In answering the questions in this paper it is particularly important to show your argumentation. The total number of points will only be given for full and detailed answers.**
4. **Electronic devices calculators, tablets, laptops and mobile phones are not allowed.**
5. **Notes and textbooks are not allowed.**
6. **Candidates are reminded of the need to use clear and accurate English.**
7. **Please write your solutions in the space provided below each problem on the accompanying pages, and ensure that your name and student ID are on every page.**

DO NOT REMOVE THIS DOCUMENT FROM THE EXAMINATION ROOM.

Att	Q1	Q2	Q3	Q4	Q5	Bonus	TOTAL
10 pts	18 pts	19 pts	15 pts	18 pts	20 pts	5 pts	100 pts

Full Name: .....

Student Number: .....

Name: **Student Number:**

1. Let G be the cyclic group of order 12 generated by g.

   (a) (9 Points) Find all subgroups of G (express their generators in terms of g).

   Solution: By the theory of cyclic groups, G has a unique subgroup of size d for every divisor d of 12, namely $H_d = \langle g^{12/d} \rangle$. The divisors of 12 are $1, 2, 3, 4, 6, 12$.

   - Let $d = 1$, then the corresponding subgroup is $H_1 = \{g^{12} = id\}$.
   - Let $d = 2$, then the corresponding subgroup is $H_2 = \langle g^{12/2} = g^6 \rangle = \{id, g^6\}$.
   - Let $d = 3$, then the corresponding subgroup is $H_3 = \langle g^{12/3} = g^4 \rangle = \{id, g^4, g^8\}$.
   - Let $d = 4$, then the corresponding subgroup is $H_4 = \langle g^{12/4} = g^3 \rangle = \{id, g^3, g^6, g^9\}$.
   - Let $d = 6$, then the corresponding subgroup is $H_6 = \langle g^{12/6} = g^2 \rangle = \{id, g^2, g^4, g^6, g^8, g^{10}\}$.
   - Let $d = 12$, then the corresponding subgroup is $H_{12} = \langle g \rangle = G$.

   ■

   (b) (9 Points) Let H be the unique subgroup of size 2 of G. Find all subgroups of G/H. (you must express generators of each subgroup in terms of cosets of H)

   Solution: By part (a), $H = H_2 = \langle g^6 \rangle = \{id, g^6\}$. Since G is cyclic, G/H is cyclic, generated by the coset gH, and $|G/H| = 12/2 = 6$. Since $g^6 \in H$, we have $g^6 H = H$, so

   $$G/H = \{H,\ gH,\ g^2H,\ g^3H,\ g^4H,\ g^5H\}.$$

   Hence G/H has a unique subgroup of size d for every divisor d of 6, namely $\langle (gH)^{6/d} \rangle = \langle g^{6/d} H \rangle$. The divisors of 6 are $1, 2, 3, 6$.

   - Let $d = 1$, then the corresponding subgroup is $K_1 = \langle (gH)^6 = g^6 H = H \rangle = \{H\}$.
   - Let $d = 2$, then the corresponding subgroup is $K_2 = \langle (gH)^3 = g^3 H \rangle = \{H, g^3 H\}$.
   - Let $d = 3$, then the corresponding subgroup is $K_3 = \langle (gH)^2 = g^2 H \rangle = \{H, g^2 H, g^4 H\}$.
   - Let $d = 6$, then the corresponding subgroup is $K_6 = \langle gH \rangle = G/H$.

   This agrees with the correspondence theorem: the subgroups of G containing H are H_2, H_4, H_6, H_{12}, and $K_1 = H_2/H$, $K_2 = H_4/H$, $K_3 = H_6/H$, $K_6 = H_{12}/H$. ■

2. (19 Points) Let G be a group of size 55, H be a group of size 20 and K be a group of size 11. It is given that there exists:

   - A nontrivial group homomorphism $\alpha : G \to H$ and
   - An onto group homomorphism $\beta : G \to K$.

   (a) (5 Points) Show that $\ker \alpha$ is a normal subgroup of G of size 11.

   Solution: We use the following theorems:
   First Isomorphism Theorem: If $\varphi : G \to H$ is a group homomorphism, then $\ker \varphi$ is a normal subgroup of G and $G/\ker \varphi \cong \varphi(G)$. In particular, $|G| = |\ker \varphi| \cdot |\varphi(G)|$.
   Lagrange's Theorem: If L is a subgroup of a finite group G, then $|L|$ divides $|G|$.
   Since $\alpha(G)$ is a subgroup of H, by Lagrange's theorem $|\alpha(G)|$ divides $|H| = 20$. By the first isomorphism theorem, $|\alpha(G)| = 55/|\ker \alpha|$, so $|\alpha(G)|$ also divides 55. Hence $|\alpha(G)|$ divides $\gcd(55, 20) = 5$, so $|\alpha(G)| \in \{1, 5\}$. Since α is nontrivial, $|\alpha(G)| \neq 1$, so $|\alpha(G)| = 5$.
   By the first isomorphism theorem, $\ker \alpha$ is a normal subgroup of G and $|\ker \alpha| = 55/5 = 11$. ■

   (b) (4 Points) Show that $\ker \beta$ is a normal subgroup of G of size 5.

   Solution: Since β is onto, $\beta(G) = K$, so $|\beta(G)| = 11$. By the first isomorphism theorem, $\ker \beta$ is a normal subgroup of G and $|\ker \beta| = 55/11 = 5$. ■

   (c) (10 Points) Show that G is cyclic.

   Solution: We use the following theorems:
   Theorem 1: If M and N are normal subgroups of a group G such that $M \cap N = \{e\}$ and $G = MN$, then $G \cong M \times N$.
   Theorem 2: If H and K are subgroups of a group G, then $|HK| = \dfrac{|H|\,|K|}{|H \cap K|}$.
   Theorem 3: $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $\gcd(m, n) = 1$. In that case, $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$.
   By parts (a) and (b), $\ker \alpha$ and $\ker \beta$ are normal subgroups of G with $|\ker \alpha| = 11$ and $|\ker \beta| = 5$. We check the conditions of Theorem 1.
   Trivial intersection: $\ker \alpha \cap \ker \beta$ is a subgroup of both $\ker \alpha$ and $\ker \beta$, so by Lagrange's theorem its order divides both 11 and 5. Hence $|\ker \alpha \cap \ker \beta| = 1$, that is, $\ker \alpha \cap \ker \beta = \{e\}$.
   $G = (\ker \alpha)(\ker \beta)$**:** By Theorem 2,

$$|(\ker \alpha)(\ker \beta)| = \frac{|\ker \alpha|\,|\ker \beta|}{|\ker \alpha \cap \ker \beta|} = \frac{11 \cdot 5}{1} = 55 = |G|.$$

   Since $(\ker \alpha)(\ker \beta) \subseteq G$ and both sets have 55 elements, $G = (\ker \alpha)(\ker \beta)$.
   Therefore, by Theorem 1,

$$G \cong \ker \alpha \times \ker \beta.$$

   Since 11 and 5 are prime, every group of prime order is cyclic, so $\ker \alpha \cong \mathbb{Z}_{11}$ and $\ker \beta \cong \mathbb{Z}_5$. Hence

$$G \cong \mathbb{Z}_{11} \times \mathbb{Z}_5.$$

   Since $\gcd(11, 5) = 1$, Theorem 3 gives $G \cong \mathbb{Z}_{55}$. Therefore G is cyclic. ■

3. (15 Points) True or False, justify your result briefly.

   (a) If G is a group such that every proper subgroup is abelian then G is abelian.

   Solution: **False.** Consider $G = S_3$, which has order 6. By Lagrange's theorem, a proper subgroup of S_3 has order 1, 2 or 3. Groups of order 1 are trivial, and groups of prime order are cyclic, so every proper subgroup of S_3 is abelian. However, S_3 is not abelian: for example,

   $$(1\ 2)(1\ 3) = (1\ 3\ 2) \neq (1\ 2\ 3) = (1\ 3)(1\ 2).$$

   ■

   (b) No group of order 33 is simple.

   Solution: **True.** Let $|G| = 33 = 3 \cdot 11$, and let n_{11} be the number of Sylow 11-subgroups of G. By the Sylow theorems, n_{11} divides 3 and $n_{11} \equiv 1 \pmod{11}$. The only possibility is $n_{11} = 1$. Hence G has a unique Sylow 11-subgroup P, which is therefore normal in G. Since $|P| = 11$, P is neither trivial nor equal to G, so G is not simple. ■

   (c) If G is free abelian group then G cannot contain any torsion element other than identity.

   Solution: **True** (apart from the identity, which has order 1). A free abelian group is isomorphic to a direct sum of copies of $\mathbb{Z}$. Let $x = (a_1, a_2, \ldots)$ be a nonzero element, so some $a_i \neq 0$. For every integer $m \geq 1$, the i-th coordinate of mx is $ma_i \neq 0$, so $mx \neq 0$. Hence no nonzero element of G has finite order, that is, G is torsion-free. ■

   (d) Let G be a group of size 14, then $1 + 1 + 1 + 1 + 1 + 2 + 7$ is a possible class equation for G.

   Solution: **False.** In the class equation, the number of 1's equals the number of conjugacy classes of size 1, which is $|Z(G)|$. Here this would give $|Z(G)| = 5$. But $Z(G)$ is a subgroup of G, so by Lagrange's theorem $|Z(G)|$ must divide 14, and $5 \nmid 14$. Hence this is not a possible class equation. ■

   (e) Let H, K, and L be normal subgroups of G with $H < K < L$. Let $A = G/H$, $B = K/H$, and $C = L/H$. Then $(A/B)/(C/B)$ is isomorphic to G/L.

   Solution: **True.** By the third isomorphism theorem,

   $$A/B = (G/H)/(K/H) \cong G/K \qquad \text{and} \qquad C/B = (L/H)/(K/H) \cong L/K,$$

   where the first isomorphism, $gH(K/H) \mapsto gK$, maps the subgroup C/B onto L/K. Hence

   $$(A/B)/(C/B) \cong (G/K)/(L/K) \cong G/L,$$

   where the last step is the third isomorphism theorem applied to $K \subseteq L$, both normal in G. ■

4. (18 Points) Let p be a prime and let G be a group of order p^n where $n \geq 1$. Let N be a normal subgroup of G of order p.

   Recall that the center of G is $Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}$.

   (a) (6 Points) Show that $g \cdot x = gxg^{-1}$ defines an action of G on the set N.

   Solution: **Well-defined:** Since N is normal in G, for every $g \in G$ and $x \in N$ we have $gxg^{-1} \in N$. So $g \cdot x \in N$.
   Identity: For every $x \in N$, $e \cdot x = exe^{-1} = x$.
   Compatibility: For $g, h \in G$ and $x \in N$,

   $$g \cdot (h \cdot x) = g(hxh^{-1})g^{-1} = (gh)x(gh)^{-1} = (gh) \cdot x.$$

   Hence this is an action of G on N. ■

   (b) (6 Points) Show that every orbit of this action has exactly one element. Deduce that $N \subseteq Z(G)$.

   Solution: We use the following theorem:
   Orbit-Stabilizer Theorem: If a finite group G acts on a set X and $x \in X$, then $|O_x| = |G|/|G_x|$.
   By the orbit-stabilizer theorem, $|O_x|$ divides $|G| = p^n$, so $|O_x|$ is a power of p for every $x \in N$.
   The orbit of e: $geg^{-1} = e$ for all $g \in G$, so $Ge = \{e\}$.
   The orbit of $x \neq e$: If $gxg^{-1} = e$, then $x = g^{-1}eg = e$. So for $x \neq e$, every element of O_x is different from e, that is, $O_x \subseteq N \setminus \{e\}$. Hence

   $$|O_x| \leq |N| - 1 = p - 1 < p.$$

   Since $|O_x|$ is a power of p and less than p, we get $|O_x| = 1$.
   So every orbit has exactly one element, that is, $gxg^{-1} = x$ for all $g \in G$ and $x \in N$. Hence $gx = xg$ for all $g \in G$, so every $x \in N$ lies in $Z(G)$. Therefore $N \subseteq Z(G)$. ■

   (c) (6 Points) Show that G has at least one normal subgroup of order p.

   Solution: We use the following theorem:
   Theorem: If G is a group of order p^n where p is prime and $n \geq 1$, then $Z(G)$ is nontrivial.
   By the theorem, $Z(G)$ is nontrivial, so p divides $|Z(G)|$. By Cauchy's theorem, $Z(G)$ contains an element w of order p, so $W = \langle w \rangle$ is a subgroup of order p with $W \subseteq Z(G)$. For every $g \in G$ and $w^i \in W$, we have $gw^ig^{-1} = w^i \in W$, since $w^i \in Z(G)$. Hence W is a normal subgroup of G of order p. ■

5. Let $G = S_5$, the symmetric group (permutation group) on $\{1, 2, 3, 4, 5\}$.

   (a) (5 Points) Show that any order 4 element of G is of the form $(a\ b\ c\ d)$ where a, b, c, d are pairwise distinct elements in $\{1, 2, 3, 4, 5\}$.

   Solution: Let $\tau \in S_5$ be an element of order 4. We use the following theorems:

   Theorem 1: Every permutation in S_n can be written uniquely (up to order) as a product of disjoint cycles.

   Theorem 2: If $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ is a product of disjoint cycles, then $|\sigma| = \text{lcm}(|\sigma_1|, |\sigma_2|, \ldots, |\sigma_k|)$, where $|\sigma_i|$ is the length of the cycle σ_i.

   Write τ as a product of disjoint cycles. The possible cycle structures of permutations in S_5 correspond to the partitions of 5. We list them with the lcm of the cycle lengths:

   - 5: $\text{lcm} = 5$
   - $4 + 1$: $\text{lcm} = 4$
   - $3 + 2$: $\text{lcm} = 6$
   - $3 + 1 + 1$: $\text{lcm} = 3$
   - $2 + 2 + 1$: $\text{lcm} = 2$
   - $2 + 1 + 1 + 1$: $\text{lcm} = 2$
   - $1 + 1 + 1 + 1 + 1$: $\text{lcm} = 1$

   The only partition giving $\text{lcm} = 4$ is $4 + 1$, corresponding to one 4-cycle and one fixed point. Therefore $\tau = (a\ b\ c\ d)$ where a, b, c, d are pairwise distinct elements of $\{1, 2, 3, 4, 5\}$. ■

   (b) (4 Points) How many order 4 elements are there in G?

   Solution: By part (a), the elements of order 4 are exactly the 4-cycles. To form a 4-cycle we first choose the 4 elements that appear in it: there are $\binom{5}{4} = 5$ ways. Given 4 elements, there are $4! = 24$ ways to list them as $(a\ b\ c\ d)$, but each 4-cycle is listed 4 times, since

   $$(a\ b\ c\ d) = (b\ c\ d\ a) = (c\ d\ a\ b) = (d\ a\ b\ c).$$

   So there are $24/4 = 6$ distinct 4-cycles on a fixed set of 4 elements. Therefore the number of elements of order 4 in G is

   $$5 \cdot 6 = 30.$$

   ■

(c) (5 Points) Let N be a normal subgroup of G containing an element of order 4. Using part (b), show that $|N| > 31$.

Solution: We use the following theorem:

Theorem: Two permutations in S_n are conjugate if and only if they have the same cycle structure.

Since N is normal, $\pi\sigma\pi^{-1} \in N$ for every $\sigma \in N$ and $\pi \in G$. Hence, by the theorem, if N contains a permutation, it contains every permutation with the same cycle structure.

By part (a), N contains a 4-cycle $\sigma = (a\ b\ c\ d)$, hence all 4-cycles. By part (b), these are 30 elements. Also $id \in N$, so $|N| \geq 31$.

Moreover, $\sigma^2 = (a\ c)(b\ d) \in N$. This element is neither the identity nor a 4-cycle, so it is different from the 31 elements above. Therefore $|N| \geq 32 > 31$. ■

(d) (6 Points) Show that $|N| > 60$ and deduce that $N = G$. (*Hint:* Can a product of two 4-cycles be a 3-cycle?)

Solution: As in part (c), N contains every permutation with the same cycle structure as one of its elements.

- By part (c), N contains all 30 4-cycles and the identity.
- By part (c), N contains an element of the form $(a\ b)(c\ d)$, hence all such elements. There are 5 choices for the fixed point and 3 ways to split the remaining 4 elements into two pairs, giving $5 \cdot 3 = 15$ elements.
- Since (1 2 3 4) and (1 3 2 4) are 4-cycles, both lie in N, so their product lies in N. Composing from right to left,

$$(1\ 2\ 3\ 4)(1\ 3\ 2\ 4) = (1\ 4\ 2).$$

  Hence N contains a 3-cycle, hence all 3-cycles. There are $\binom{5}{3} \cdot 2 = 20$ of them.

These elements have different cycle structures, so they are distinct. Therefore

$$|N| \geq 30 + 1 + 15 + 20 = 66 > 60.$$

By Lagrange's theorem, $|N|$ divides $|G| = 5! = 120$. The only divisor of 120 greater than 60 is 120 itself, so $|N| = 120 = |G|$. Hence $N = G$. ■

Name: **Student Number:**

BONUS (5 Points:) Write your favorite result from this class, you do not need to prove it.